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A linear algorithm for the group path problem on chordal graphs

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Abstract

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Assume that each edge of a graph $G=(V,E)$ is given a weight, which is an element of some group $\mathscr{G}$. The weight of a path P is defined as the product of the weights of the edges along P. The *group path problem* is to find a chordless path of a given weight between two given vertices. It generalizes the parity path problem considered by Hsu. We show that the recognition problem associated with the group path problem is NP-complete in general, and present an $O(|\mathscr{G}|\cdot|E|+|V|)$ time algorithm for the group path problem on a chordal graph.

Keywords. Parity path problem, chordal graphs, graph algorithms.

1. Introduction

We consider a finite, undirected, and simple graph $G=(V,E)$. A *chord* of a path or a cycle in G is an edge between two nonconsecutive vertices along the path or cycle. We assume that each edge $e\in E$ is given a *weight* $\omega(e)$ taken from some group $\mathscr{G}$. The weight of a path is defined as the product of its edge weights taken along the path, the weight of a path of length zero being the identity element of the group. Given two vertices s and t, and a group element g, the *group path problem* is to find a chordless path of weight g from s to t, if one exists. When $\mathscr{G}$ is the additive group mod 2 and each edge has a unit weight, the problem becomes the parity path problem considered in [6] in the context of recognizing planar perfect graphs. The

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graph $\mathcal{G}$ is called *chordal* if every cycle of length >3 has a chord. Chordal graphs form an important class of perfect graphs, and are thoroughly explored [3, 5]. Recently, efficient parallel algorithms for recognition and some optimization problems on chordal graphs have been proposed in [7]. Below we show that the recognition problem associated with the group path problem is NP-complete in general, and present an $O(|\mathcal{G}| \cdot |E| + |V|)$ time algorithm for the group path problem on chordal graphs, assuming constant time per group operation.

2. NP-completeness

In this section, the group path problem is shown to be NP-complete by a polynomial reduction from the Hamiltonian path problem between a pair of specified vertices. The two problems are formulated as follows:

GROUP PATH.

Instance: A graph $G=(V,E)$; vertices $s,t\in V$; a group $\mathcal{G}$; a weight function $\omega : E \to \mathcal{G}$; a group element $g \in \mathcal{G}$.

Question: Does G have a chordless path from s to t of weight g?

HAMILTONIAN PATH.

Instance: A graph G and two specified vertices s and t.

Question: Does G have a Hamiltonian path from s to t?

This is a well-known NP-complete problem [4].

Theorem 2.1. *GROUP PATH is* NP-*complete.*

Proof. Let $G=(V,E)$ be an instance of HAMILTONIAN PATH. Construct a graph $G'=(V',E')$ as follows: subdivide every edge of G into two edges in series, i.e., $V'=V\cup U$, where $U=\{u_e : e\in E\}$, and $E'=\{u_e x : x \text{ is an endpoint of } e\in E\}$. So G' has $|V|+|E|$ vertices and $2|E|$ edges.

It is easy to see that G' is bipartite and every path in G' between vertices of V is chordless. It is also clear that for $x,y\in V$, G has a path of length k from x to y if and only if G' has a path of length $2k$ from x to y. So G contains a Hamiltonian path from s to t if and only if G' contains a chordless path of length $2(|V|-1)$ from s to t.

We complete the reduction by exhibiting a group $\mathcal{G}$, weights for the edges of G', and a group element g such that a path in G' from s to t has length $2|V|-2$ if and only if it has weight g. Take $\mathcal{G}$ to be the cyclic group $\mathbb{Z}_{2|V|-1}$, give unit weights to the edges of G', and let $g=2|V|-2$. Note that the length of every path in G' from s to t is at most $2|V|-2$, and since its weight equals its length mod $(2|V|-1)$, the weight equals the length. This proves the required equivalence. □

Remark. The above reduction, in fact, proves that GROUP PATH remains NP-complete even for bipartite graphs and cyclic groups. The following restriction of the problem, which is closely related to the parity path problem, is shown to be NP-complete in [2]: Given a graph, does there exist an odd chordless path between a pair of specified vertices?

3. Main result

A *perfect elimination order* (PEO) of a graph $G=(V,E)$ is a bijective labelling $l: V \to \{1,\dots,|V|\}$ such that for any three vertices $u, v, w \in V$, if $uv, uw \in E$, $l(u) < l(v)$, $l(u) < l(w)$, then $vw \in E$. We denote the latter property by PEO$[u,v,w]$ for easy reference.

Elimination orderings arise in the study of Gaussian elimination on sparse matrices. The theory of elimination orderings is intimately related to the theory of clique separators. See [8] for details.

The following characterization of chordal graphs is well known (see [5]).

Theorem 3.1. *A graph $G=(V,E)$ is chordal if and only if G has a PEO. Moreover, in that case, for every vertex v there is a PEO l such that $l(v)=|V|$. One can recognize a chordal graph and find such a PEO in linear time* $O(|V|+|E|)$.

From now on, we assume that a PEO l is given for G, and identify each vertex v with its label $l(v)$.

A path P with vertices $v_1,\dots,v_r$ in this order will be denoted by $P: v_1 \cdots v_r$. Such a path is said to be *normalized* if $v_i < v_r$ for $1 \le i \le r-1$. The following simple lemma has many algorithmic consequences for chordless paths in chordal graphs.

Lemma 3.2. *If $P: v_1 \cdots v_r$ is a normalized chordless path, then $v_1 < v_2 < \cdots < v_r$.*

Proof. We prove $v_i < v_{i+1}$ for $i=1,\dots,r-1$ by induction on $r-1-i$. The basis $r-1-i=0$ follows from the definition of a normalized path. If $v_{i+1} < v_{i+2}$ but $v_{i+1} < v_i$, then from PEO$[v_{i+1}, v_i, v_{i+2}]$ we get $v_i v_{i+2} \in E$, so P has a chord, a contradiction. Therefore $v_i < v_{i+1}$. □

From Lemma 3.2 it follows that each subpath of a normalized chordless path is itself a normalized chordless path. This allows for the construction of a normalized path in an incremental fashion.

Our problem is to find a chordless path of weight g from s to t in a chordal graph G. By Theorem 3.1 we may assume that $l(t)=|V|$, so that every path leading to t is automatically normalized. So we may as well seek a normalized chordless path of weight g from s to t. We call a normalized chordless path $P: s=v_1 \cdots v_r = v$ of weight g from s to any vertex v a g-path, and denote the predecessor v_{r-1} of v

along P by $\mathrm{pred}_P(v)$, if it exists, that is if $r>1$. For any $v \neq s$ and any group element g, we define the g-set of v by

$$g\text{-set}(v)=\{u\in V\colon\ u<v,\ uv\in E, \text{ and there exists an } h\text{-path } Q \text{ to } u \text{ satisfying } \mathrm{pred}_Q(u)v\notin E \text{ if } \mathrm{pred}_Q(u) \text{ exists, where } h=g\omega^{-1}(uv)\}.$$

The vertices of g-set(v) are the natural candidates for the penultimate vertex on a g-path to v. This is established by the following lemma.

Lemma 3.3. *Let $v \neq s$ and let g be a group element.*

(i) *If P is a g-path to v, then* $\mathrm{pred}_P(v)\in g$-set(v).

(ii) *If $u\in g$-set(v), then there exists a g-path P to v via u, i.e.,* $u=\mathrm{pred}_P(v)$.

Proof. Part (i) follows directly from the definitions. To prove part (ii), consider the h-path $Q: s=v_1v_2\cdots v_r=u$ to u, guaranteed by the definition of g-set(v). Certainly the path $P: s=v_1v_2\cdots v_rv$ has weight g and is normalized. To show that P is chordless, it suffices to show that $v_iv\notin E$ for $i=1,\ldots,r-1$. Again, we use induction on $r-1-i$. The basis $r-1-i=0$ follows from the definition of Q. If $v_{i+1}v\notin E$, then $v_iv\notin E$, for otherwise PEO$[v_i, v_{i+1}, v]$ is contradicted. □

It is easy to characterize $s\in g$-set(v) by $s<v$, $sv\in E$, and $\omega(sv)=g$. Theorem 3.4 below characterizes $u\in g$-set(v) recursively in the remaining case $s<u<v$. We denote by g-min(v) the minimum of g-set(v), the minimum of the empty set being ∞.

Theorem 3.4. *Assume $s<u<v$ and $uv\in E$, and let g be a group element. Then $u\in g$-set(v) if and only if*

(1) *h-min(u)$\neq\infty$, and*

(2) *h-min(u)$v\notin E$,*

where $h=g\omega^{-1}(uv)$.

Proof. *If*: Put $x=h$-min(u). By (1) and Lemma 3.3(ii), there is an h-path to u via x, and so by (2) and the definition of g-set(v), $u\in g$-set(v).

Only if: Since $u\in g$-set(v) and by Lemma 3.3(ii), there exists a g-path $P: s=v_1\cdots v_{p-1}v_p$, where $v_{p-1}=u$ and $v_p=v$. We have $p\geq 3$, since $s<u<v$. Also, since the subpath $v_1\cdots v_{p-1}$ is an h-path to u, it follows from Lemma 3.3(i) that $v_{p-2}\in h$-set(u). Therefore $w=h$-min(u)$\neq\infty$, and it remains to show that $wv\notin E$. We shall assume $wv\in E$ and derive a contradiction. Note that w is not on P (because P is chordless), and therefore $w<v_{p-2}<u<v$. Note also that $wu\in E$, because $w\in h$-set(u).

We assert that $wv_i\notin E$ for $i=1,\ldots,p-2$. The proof of the assertion is by induction on $p-2-i$. For the basis $i=p-2$, $wv_{p-2}\in E$ would imply that P has the chord $v_{p-2}v$ by PEO$[w, v_{p-2}, v]$, hence $wv_{p-2}\notin E$. For the induction step, if

$wv_{i+1}, \ldots, wv_{p-2} \notin E$, then $wv_i \notin E$, for otherwise G contains a chordless cycle $wv_i v_{i+1} \cdots v_{p-2}u$ of length ≥ 4. This proves the assertion.

Since $w \in h$-set(u), it follows from Lemma 3.3(ii) that there exists an h-path $Q: s = u_1 \cdots u_{q-1}u_q$, where $u_{q-1} = w$ and $u_q = u$. Since $s \leq w < u$, $q \geq 2$, but in fact $q \geq 4$; for if $q = 2$, then $w = s$, but s is on P and w is not, and if $q = 3$ then w is adjacent to $u_1 = v_1$, contrary to the assertion. It also follows that $p \geq 4$, for if $p = 3$, then uv_1 is a chord of Q.

Let k be the largest integer such that $k \leq p-2$ and v_k is adjacent to one of $u_2, \ldots, u_{q-2}$ (k exists since $1 \leq p-2$ and $v_1 = u_1 u_2 \in E$). Also, let l be the largest integer such that $l \leq q-1$ and $u_l v_k \in E$. The assertion implies that, in fact, $l \leq q-2$. Now G contains the chordless cycle $u_l \cdots u_{q-2} w u v_{p-2} \cdots v_k$ of length ≥ 4 (see Fig. 1), a contradiction. □

To implement the results of Theorem 3.4, trace back a g_0-path to t using values g-min(v) computed recursively by minimizing h-min(u) among all u such that $s < u < v$, $uv \in E$, $wv \notin E$, where $h = g\omega^{-1}(uv)$ and $w = h$-min(u). All this can be done in time $O(|\mathscr{G}| \cdot |E| + |V|)$ if G is represented by its adjacency lists; this time

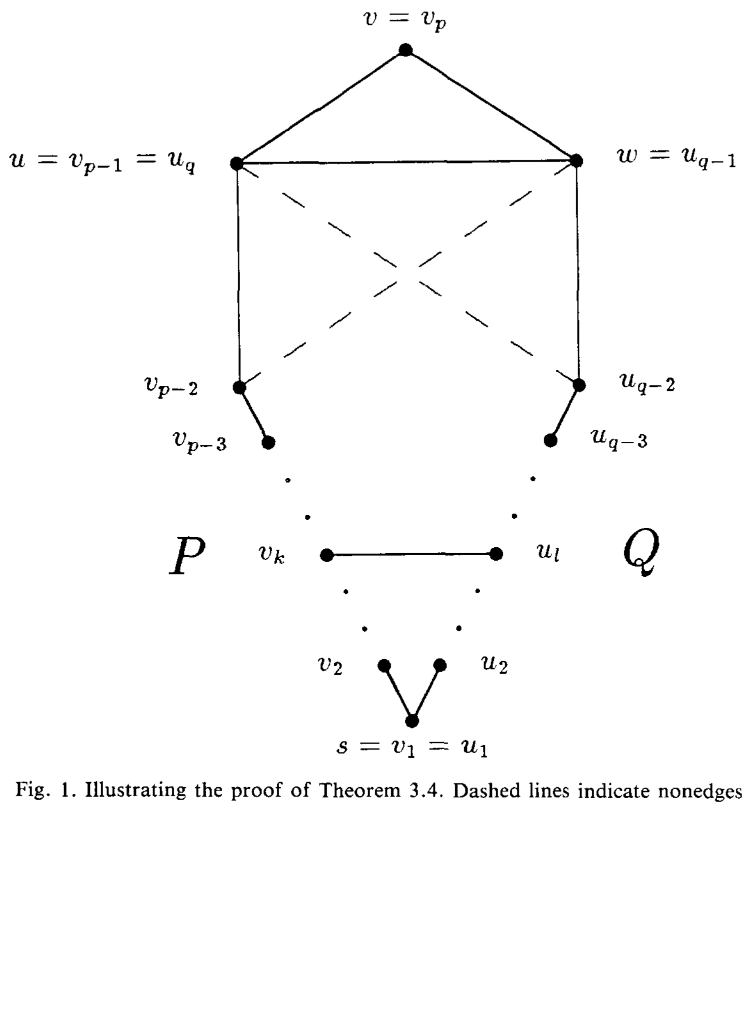

Fig. 1. Illustrating the proof of Theorem 3.4. Dashed lines indicate nonedges.

is linear if the group $\mathscr{G}$ is fixed. When a vertex v is visited, the corresponding row of the adjacency matrix is constructed in time $O(\deg v)$, and used to check for $uv \in E$, $wv \notin E$ in constant time. For the details, see [1].

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